

# Well-posedness, global existence and blow-up phenomena for an integrable multi-component Camassa-Holm system

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## Abstract

This paper is concerned with a multi-component Camassa-Holm system, which has been proven to be integrable and has peakon solutions. This system includes many one-component and two-component Camassa-Holm type systems as special cases. In this paper, we first establish the local well-posedness and a continuation criterion for the system, then we present several global existence or blow-up results for two important integrable two-component subsystems. Our obtained results cover and improve recent results in [25, 36].

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## 1 Introduction

In this paper, we consider the following multi-component system proposed by Xia and Qiao in [34]:

$$(1.1) \quad \begin{cases} m_{jt} = (m_j H)_x + m_j H + \frac{1}{(N+1)^2} \sum_{i=1}^N [m_i(u_j - u_{jx})(v_i + v_{ix}) + m_j(u_i - u_{ix})(v_i + v_{ix})], \\ n_{jt} = (n_j H)_x - n_j H - \frac{1}{(N+1)^2} \sum_{i=1}^N [n_i(u_i - u_{ix})(v_j + v_{jx}) + n_j(u_i - u_{ix})(v_i + v_{ix})], \\ m_j = u_j - u_{jxx}, n_j = v_j - v_{jxx}, 1 \leq j \leq N, \end{cases}$$

where  $H$  is an arbitrary function of  $u_j, v_j$ ,  $1 \leq j \leq N$ , and their derivatives. The above  $2N$ -component Camassa-Holm system is proved to be integrable in the sense of Lax pair and infinitely many conservation laws in [34], where its peakon solutions for the case  $N = 2$  are also obtained.

Since  $H$  is an arbitrary function of  $u_j, v_j$ ,  $1 \leq j \leq N$ , and their derivatives, thus Eq.(1.1) is actually a large class of systems. As  $N = 1$ ,  $v_1 = 2$  and  $H = -u_1$ , Eq.(1.1) is reduced to the standard Camassa-Holm (CH) equation

$$(1.2) \quad m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx},$$

which was derived by Camassa and Holm [4] in 1993 as a model for the unidirectional propagation of shallow water waves over a flat bottom. The CH equation, also as a model for the propagation of axially symmetric waves in hyperelastic rods [17], has a bi-Hamiltonian structure [7, 22] and is completely integrable [4, 6]. One of the remarkable properties of the CH equation is the existence of peakons. One can refer to [1, 4, 14, 15, 16] for the existence of peakon solitons and multi-peakons. The Cauchy problem and initial boundary

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problem of the CH equation has been studied extensively: local well-posedness [8, 11, 18, 26, 31, 19, 20], global strong solutions [5, 8, 11, 19, 20], blow-up solutions in finite time [5, 8, 10, 12, 27, 19, 20] and global weak solutions [3, 9, 13, 35].

As  $N = 1$  and  $H = -\frac{1}{2}(u_1 - u_{1x})(v_1 + v_{1x})$ , Eq.(1.1) is reduced to the following system proposed by Song, Qu and Qiao in [32]:

$$(1.3) \quad \begin{cases} m_t + \frac{1}{2}((u - u_x)(v + v_x)m)_x = 0, \\ n_t + \frac{1}{2}((u - u_x)(v + v_x)n)_x = 0. \end{cases}$$

The above system is proved to be integrable not only in the sense of Lax-pair but also in the sense of geometry, namely, it describes pseudospherical surfaces [32]. Besides, exact solutions to this system such as cuspons and W/M-shape solitons are also obtained in [32].

As  $N = 1$  and  $H = -\frac{1}{2}(u_1v_1 - u_{1x}v_{1x})$ , Eq.(1.1) is reduced to the following system proposed by Xia and Qiao in [30, 33]:

$$(1.4) \quad \begin{cases} m_t + \frac{1}{2}((uv - u_xv_x)m)_x - \frac{1}{2}(uv_x - vu_x)m = 0, \\ n_t + \frac{1}{2}((uv - u_xv_x)n)_x + \frac{1}{2}(uv_x - vu_x)n = 0, \end{cases}$$

which describes a nontrivial one-parameter family of pseudo-spherical surfaces. In [30, 33], the authors showed this system is integrable with Lax pair, bi-Hamiltonian structure, and infinitely many conservation laws. They also studied the peaked soliton and multi-peakon solutions to the system. Recently, Yan, Qiao and Yin [36] studied the local well-posedness for the Cauchy problem of the system and derived a precise blow-up scenario and a blow-up result for the strong solutions to the system.

As  $v = 2u$ , both Eq.(1.3) and Eq.(1.4) are reduced to the following cubic Camassa-Holm equation

$$(1.5) \quad m_t + (m(u^2 - u_x^2))_x = 0, \quad m = u - u_{xx},$$

which was proposed independently by Fokas [21], Fuchssteiner [24], Olver and Rosenau [28], and Qiao [29] as an integrable peakon equations with cubic nonlinearity. Its Lax pair, peakon and soliton solutions, local well-posedness and blow-up phenomena have been studied in [29, 23, 25].

The aim of this paper is to establish the local well-posedness and a continuation criterion for the Cauchy problem of Eq.(1.1) in Besov spaces, and present several global existence or blow-up results for the two component subsystems: Eq.(1.3) and Eq.(1.4). Our obtained results cover and improve recent results in [25, 36]. Compared with the Camassa-Holm equation, one of the remarkable features of Eq.(1.1) is that it has higher-order nonlinearities. Thus, we have to estimate elaborately these higher-order nonlinear terms for the study of the local well-posedness and the continuation criterion of Eq.(1.1) in Besov spaces.

Besides, we derive that  $\|m(t)\|_{L^1}$  ( $\|n(t)\|_{L^1}$ ) and  $\int_{\mathbb{R}}(mv_x)(t,x)dx = \int_{\mathbb{R}}(nu_x)(t,x)dx$  are conservation laws for Eq.(1.3) and Eq.(1.4), respectively. The above conservation laws, which have not been derived or used in the associated previous papers [25, 36], are useful and crucial in some blow-up results stated in the following fourth section.

The rest of our paper is then organized as follows. In Section 2, we recall the Littlewood-Paley decomposition and some basic properties of the Besov spaces. In Section 3, we establish the local well-posedness and provide a continuation criterion for Eq.(1.1). The last section is devoted to establishing several global existence or blow-up results for Eq.(1.3) and Eq.(1.4).

From now on we always assume that  $H = H(u_1, \dots, u_N, v_1, \dots, v_N, u_{1x}, \dots, u_{1x}, v_{1x}, \dots, v_{1x})$  is a polynomial of degree  $l$ ,  $C > 0$  stands for a generic constant,  $A \lesssim B$  denotes the relation  $A \leq CB$ . Since all function spaces in this paper are over  $\mathbb{R}$ , for simplicity, we drop  $\mathbb{R}$  in the notations of function spaces if there is no ambiguity.

## 2 Preliminaries

To begin with, we introduce the Littlewood-Paley decomposition.

**Lemma 2.1.** [2] *Let  $\mathcal{C} = \{\xi \in \mathbb{R}, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  be an annulus. There exist radial functions  $\chi$  and  $\varphi$  valued in the interval  $[0, 1]$ , belonging respectively to  $\mathcal{D}(B(0, \frac{4}{3}))$  and  $\mathcal{D}(\mathcal{C})$ , such that*

$$\forall \xi \in \mathbb{R}, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1.$$

The nonhomogeneous dyadic blocks  $\Delta_j$  and the nonhomogeneous low-frequency cut-off operator  $S_j$  are then defined as follows:

$$\begin{aligned} \Delta_j u &= 0 \quad \text{if } j \leq -2, & \Delta_{-1} u &= \chi(D)u, \\ \Delta_j u &= \varphi(2^{-j}D)u \quad \text{if } j \geq 0, & S_j u &= \sum_{j' \leq j-1} \Delta_{j'} u \quad \text{for } j \in \mathbb{Z}. \end{aligned}$$

**Definition 2.1.** [2] *Let  $s \in \mathbb{R}$  and  $(p, r) \in [1, \infty]^2$ . The nonhomogeneous Besov space  $B_{p,r}^s$  consists of all  $u \in \mathcal{S}'(\mathbb{R})$  such that*

$$\|u\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \left\| (2^{js} \|\Delta_j u\|_{L^p})_{j \in \mathbb{Z}} \right\|_{l^r(\mathbb{Z})} < \infty.$$

Let us give some classical properties of the Besov spaces.

**Lemma 2.2.** [2] The set  $B_{p,r}^s$  is a Banach space, and satisfies the Fatou property, namely, if  $(u_n)_{n \in \mathbb{N}}$  is a bounded sequence of  $B_{p,r}^s$ , then an element  $u$  of  $B_{p,r}^s$  and a subsequence  $u_{\psi(n)}$  exist such that

$$\lim_{n \rightarrow \infty} u_{\psi(n)} = u \text{ in } \mathcal{S}' \quad \text{and} \quad \|u\|_{B_{p,r}^s} \leq C \liminf_{n \rightarrow \infty} \|u_{\psi(n)}\|_{B_{p,r}^s}.$$

**Lemma 2.3.** [2] Let  $m \in \mathbb{R}$  and  $f$  be an  $S^m$ -multiplier (i.e.  $f : \mathbb{R} \rightarrow \mathbb{R}$  is smooth and satisfies that for each multi-index  $\alpha$ , there exists a constant  $C_\alpha$  such that  $|\partial^\alpha f(\xi)| \leq C_\alpha(1 + |\xi|)^{m-|\alpha|}, \forall \xi \in \mathbb{R}$ ). Then the operator  $F(D)$  is continuous from  $B_{p,r}^s$  to  $B_{p,r}^{s-m}$ .

**Lemma 2.4.** [18] (i) For  $s > 0$  and  $1 \leq p, r \leq \infty$ , there exists  $C = C(d, s)$  such that

$$(2.1) \quad \|uv\|_{B_{p,r}^s} \leq C(\|u\|_{L^\infty} \|v\|_{B_{p,r}^s} + \|v\|_{L^\infty} \|u\|_{B_{p,r}^s}).$$

(ii) If  $1 \leq p, r \leq \infty$ ,  $s_1 \leq \frac{1}{p}$ ,  $s_2 > \frac{1}{p}$  ( $s_2 \geq \frac{1}{p}$ , if  $r = 1$ ) and  $s_1 + s_2 > \max\{0, \frac{2}{p} - 1\}$ , there exists  $C = C(s_1, s_2, p, r)$  such that

$$(2.2) \quad \|uv\|_{B_{p,r}^{s_1}} \leq C\|u\|_{B_{p,r}^{s_1}}\|v\|_{B_{p,r}^{s_2}}.$$

**Lemma 2.5.** [2, 18] Let  $1 \leq p \leq p_1 \leq \infty$ ,  $1 \leq r \leq \infty$ ,  $s > -\min\{\frac{1}{p_1}, 1 - \frac{1}{p}\}$ . Assume  $f_0 \in B_{p,r}^s, F \in L^1(0, T; B_{p,r}^s)$ ,  $v \in L^\rho(0, T; B_{\infty,\infty}^{-M})$  for some  $\rho > 1$  and  $M > 0$ , and

$$\begin{aligned} \partial_x v &\in L^1(0, T; B_{p_1,\infty}^{\frac{1}{p_1}} \cap L^\infty), & \text{if } s < 1 + \frac{1}{p_1}, \\ \partial_x v &\in L^1(0, T; B_{p_1,r}^{s-1}), & \text{if } s > 1 + \frac{1}{p_1}, \text{ or } s = 1 + \frac{1}{p_1} \text{ and } r = 1. \end{aligned}$$

Then the following transport equation

$$(2.3) \quad \begin{cases} \partial_t f + v \cdot \nabla f = F \\ f|_{t=0} = f_0, \end{cases}$$

has a unique solution  $f \in C([0, T]; B_{p,r}^s)$ , if  $r < \infty$ , or  $f \in L^\infty(0, T; B_{p,r}^s) \cap \left(\bigcap_{s' < s} C([0, T]; B_{p,r}^{s'})\right)$ , if  $r = \infty$ .

Moreover, the following inequality holds true:

$$(2.4) \quad \|f(t)\|_{B_{p,r}^s} \leq \|f_0\|_{B_{p,r}^s} + \int_0^t \|F(\tau)\|_{B_{p,r}^s} d\tau + C \int_0^t V_{p_1}'(\tau) \|f(\tau)\|_{B_{p,r}^s} d\tau$$

with

$$(2.5) \quad V_{p_1}'(t) = \begin{cases} \|\partial_x v(t)\|_{B_{p_1,\infty}^{\frac{1}{p_1}} \cap L^\infty}, & \text{if } s < 1 + \frac{1}{p_1}, \\ \|\partial_x v(t)\|_{B_{p_1,r}^{s-1}}, & \text{if } s > 1 + \frac{1}{p_1} \text{ or } s = 1 + \frac{1}{p_1}, r = 1. \end{cases}$$

### 3 Local well-posedness

In this section, we study the local well-posedness for Eq.(1.1).

To begin with, noticing  $(1 - \partial_x^2)^{-1} = \frac{1}{2}e^{-|x|}*$ , we have the following inequalities which will be frequently used in the sequel:

$$\begin{aligned} \|u\|_{B_{p,r}^s} &= \|(1 - \partial_x^2)^{-1}m\|_{B_{p,r}^s} \approx \|m\|_{B_{p,r}^{s+2}}, \quad \forall s \in \mathbb{R}, 1 \leq p, r \leq \infty. \\ \|u\|_{L^\infty} &= \left\| \frac{1}{2}e^{-|x|} * m \right\|_{L^\infty} \leq \|m\|_{L^\infty}, \\ \|u_x\|_{L^\infty} &= \left\| \frac{1}{2}(-\text{sign}(x)e^{-|x|}) * m \right\|_{L^\infty} \leq \|m\|_{L^\infty}, \\ \|u_{xx}\|_{L^\infty} &= \|u - m\|_{L^\infty} \leq 2\|m\|_{L^\infty}, \end{aligned}$$

where  $m = u - u_{xx}$ .

We now rewrite Eq.(1.1) as follows:

$$(3.1) \quad \begin{cases} M_t = H(U, U_x)M_x + A(H, H_x)M + B(U, U_x)M, \\ M|_{t=0} = M_0, \end{cases}$$

where  $M = (m_1, \dots, m_N, n_1, \dots, n_N)^T$ ,  $M_0 = (m_{10}, \dots, m_{N0}, n_{10}, \dots, n_{N0})^T$ ,  $U = (u_1, \dots, u_N, v_1, \dots, v_N)^T$ ,  $H = H(U, U_x)$  is a polynomial of degree  $l$ , and

$$A(H, H_x) = \begin{pmatrix} H_x I_{N \times N} + H I_{N \times N} & 0 \\ 0 & H_x I_{N \times N} - H I_{N \times N} \end{pmatrix}, \quad B(U, U_x) = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}$$

with

$$B_{11} = \frac{1}{(N+1)^2} \begin{pmatrix} (u_1 - u_{1x})(v_1 + v_{1x}) & \cdots & (u_1 - u_{1x})(v_N + v_{Nx}) \\ \vdots & \ddots & \vdots \\ (u_N - u_{Nx})(v_1 + v_{1x}) & \cdots & (u_N - u_{Nx})(v_N + v_{Nx}) \end{pmatrix} + \sum_{i=1}^N [(u_i - u_{ix})(v_i + v_{ix})] I_{N \times N},$$

and

$$B_{11} = -\frac{1}{(N+1)^2} \begin{pmatrix} (u_1 - u_{1x})(v_1 + v_{1x}) & \cdots & (u_N - u_{Nx})(v_1 + v_{1x}) \\ \vdots & \ddots & \vdots \\ (u_1 - u_{1x})(v_N + v_{Nx}) & \cdots & (u_N - u_{Nx})(v_N + v_{Nx}) \end{pmatrix} - \sum_{i=1}^N [(u_i - u_{ix})(v_i + v_{ix})] I_{N \times N}.$$

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### 3.1. Local existence and uniqueness

**Theorem 3.1.** *Let  $1 \leq p, r \leq \infty$ ,  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$ , and  $M_0 \in B_{p,r}^s$ . Then exists a time  $T > 0$  such that Eq.(3.1) has a unique solution  $M \in L^\infty(0, T; B_{p,r}^s) \cap E_{p,r}^s(T)$  with*

$$E_{p,r}^s(T) \triangleq \begin{cases} C([0, T]; B_{p,r}^s) \cap C^1([0, T]; B_{p,r}^{s-1}), & \text{if } r < \infty, \\ \bigcap_{s' < s} \left( C([0, T]; B_{p,r}^{s'}) \cap C^1([0, T]; B_{p,r}^{s'-1}) \right), & \text{if } r = \infty. \end{cases}$$

The proof relies heavily on the following lemma.

**Lemma 3.1.** *Let  $1 \leq p, r \leq \infty$  and  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$ . Suppose that  $M^1$  and  $M^2$  are two solutions of the Eq.(3.1) with the initial data  $M_0^1, M_0^2 \in L^\infty(0, T; B_{p,r}^s) \cap C([0, T]; \mathcal{S}')$ . Let  $M^{12} = M^1 - M^2$ ,  $U^{12} = U^1 - U^2$ , and  $q = \max\{l, 2\}$  (where  $l$  is the polynomial order of  $H$ ). Then, for all  $t \in [0, T]$ , we have*

(1) *if  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$ , but  $s \neq 2 + \frac{1}{p}$ , then*

$$(3.2) \quad \|M^{12}(t)\|_{B_{p,r}^{s-1}} \leq \|M_0^{12}\|_{B_{p,r}^{s-1}} e^{C \int_0^t (\|M^1(\tau)\|_{B_{p,r}^s}^q + \|M^2(\tau)\|_{B_{p,r}^s}^q + 1) d\tau};$$

(2) *if  $s = 2 + \frac{1}{p}$ , then*

$$(3.3) \quad \|M^{12}(t)\|_{B_{p,r}^{s-1}} \leq \|M_0^{12}\|_{B_{p,r}^{s-1}}^\theta (\|M^1(t)\|_{B_{p,r}^s} + \|M^2(t)\|_{B_{p,r}^s})^{1-\theta} e^{\theta C \int_0^t (\|M^1(\tau)\|_{B_{p,r}^s}^q + \|M^2(\tau)\|_{B_{p,r}^s}^q + 1) d\tau},$$

where  $\theta \in (0, 1)$ .

Proof. Let  $H^i = H(U^i, U_x^i)$ ,  $A^i = A(H^i, H_x^i)$ ,  $B^i = B(U^i, U_x^i)$ ,  $i = 1, 2$ , and  $H^{12} = H^1 - H^2$ ,  $A^{12} = A^1 - A^2$ ,  $B^{12} = B^1 - B^2$ . It is obvious that  $M^{12}$  solves the following transport equation

$$M_t^{12} - H^1 M_x^{12} = F_1 + F_2 + F_3$$

where  $F_1 = H^{12} M_x^2$ ,  $F_2 = (A^1 + B^1) M^{12}$  and  $F_3 = (A^{12} + B^{12}) M^2$ .

We claim that for all  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$ , we have

$$(3.4) \quad \|uv\|_{B_{p,r}^{s-1}} \lesssim \|u\|_{B_{p,r}^{s-1}} \|v\|_{B_{p,r}^s}.$$

Indeed, if  $s > 1 + \frac{1}{p}$ , then  $B_{p,r}^{s-1}$  is an algebra. Thus we have

$$\|uv\|_{B_{p,r}^{s-1}} \lesssim \|u\|_{B_{p,r}^{s-1}} \|v\|_{B_{p,r}^{s-1}} \lesssim \|u\|_{B_{p,r}^{s-1}} \|v\|_{B_{p,r}^s}.$$

On the other hand, if  $\max\{1 - \frac{1}{p}, \frac{1}{p}\} < s \leq 1 + \frac{1}{p}$ , then applying Lemma 2.4 (ii) with  $s_1 = s - 1$  and  $s_2 = s$  yields (3.4).

Therefore, for all  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$ , noticing the fact that  $B_{p,r}^s$  is an algebra, one may infer the following inequalities:

$$\begin{aligned}
\|F_1\|_{B_{p,r}^{s-1}} &\lesssim \|H^{12}\|_{B_{p,r}^s} \|M_x^2\|_{B_{p,r}^{s-1}} \\
&\lesssim (\|U^{12}\|_{B_{p,r}^s} + \|U_x^{12}\|_{B_{p,r}^s})(\|U^1\|_{B_{p,r}^s}^{l-1} + \|U_x^1\|_{B_{p,r}^s}^{l-1} + \|U^2\|_{B_{p,r}^s}^{l-1} + \|U_x^2\|_{B_{p,r}^s}^{l-1}) \|M_x^2\|_{B_{p,r}^{s-1}} \\
&\lesssim \|M^{12}\|_{B_{p,r}^{s-1}} (\|M^1\|_{B_{p,r}^s}^q + \|M^2\|_{B_{p,r}^s}^q + 1), \\
\|F_2\|_{B_{p,r}^{s-1}} &\lesssim (\|A^1\|_{B_{p,r}^s} + \|B^1\|_{B_{p,r}^s}) \|M^{12}\|_{B_{p,r}^{s-1}} \\
&\lesssim (\|U^1\|_{B_{p,r}^s}^l + \|U_x^1\|_{B_{p,r}^s}^l + \|U_{xx}^1\|_{B_{p,r}^s}^l + \|U^1\|_{B_{p,r}^s}^2 + \|U_x^1\|_{B_{p,r}^s}^2 + 1) \|M^{12}\|_{B_{p,r}^{s-1}} \\
&\lesssim \|M^{12}\|_{B_{p,r}^{s-1}} (\|M^1\|_{B_{p,r}^s}^q + \|M^2\|_{B_{p,r}^s}^q + 1), \\
\|F_3\|_{B_{p,r}^{s-1}} &\lesssim (\|A^{12}\|_{B_{p,r}^{s-1}} + \|B^{12}\|_{B_{p,r}^{s-1}}) \|M^2\|_{B_{p,r}^s} \\
&\lesssim (\|U^{12}\|_{B_{p,r}^{s-1}} + \|U_x^{12}\|_{B_{p,r}^{s-1}} + \|U_{xx}^{12}\|_{B_{p,r}^{s-1}})(\|U^1\|_{B_{p,r}^s}^{l-1} + \|U_x^1\|_{B_{p,r}^s}^{l-1} + \|U_{xx}^1\|_{B_{p,r}^s}^{l-1} + \|U^2\|_{B_{p,r}^s}^{l-1} \\
&\quad + \|U_x^2\|_{B_{p,r}^s}^{l-1} + \|U_{xx}^2\|_{B_{p,r}^s}^{l-1} + \|U^1\|_{B_{p,r}^s} + \|U_x^1\|_{B_{p,r}^s} + \|U^2\|_{B_{p,r}^s} + \|U_x^2\|_{B_{p,r}^s}) \|M^2\|_{B_{p,r}^s} \\
&\lesssim \|M^{12}\|_{B_{p,r}^{s-1}} (\|M^1\|_{B_{p,r}^s}^q + \|M^2\|_{B_{p,r}^s}^q + 1),
\end{aligned}$$

with  $q = \max\{l, 2\}$ .

Thus, for the case (1)  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$  and  $s \neq 2 + \frac{1}{p}$ , using Lemma 2.5 with the above three inequalities and  $p_1 = p$  and

$$V_{p_1}'(t) = \|\partial_x H^1\|_{B_{p,r}^{s-2}} + \|\partial_x H^1\|_{B_{p,r}^{\frac{1}{p}} \cap L^\infty} \leq \|\partial_x H^1\|_{B_{p,r}^s},$$

we have

$$\begin{aligned}
(3.5) \quad \|M^{12}(t)\|_{B_{p,r}^{s-1}} &\leq \|M_0^{12}\|_{B_{p,r}^{s-1}} + \int_0^t \|(F_1 + F_2 + F_3)(\tau)\|_{B_{p,r}^{s-1}} d\tau + C \int_0^t V_{p_1}'(\tau) \|M^{12}(\tau)\|_{B_{p,r}^{s-1}} d\tau \\
&\leq \|M_0^{12}\|_{B_{p,r}^{s-1}} + C \int_0^t \|M^{12}(\tau)\|_{B_{p,r}^{s-1}} (\|M^1(\tau)\|_{B_{p,r}^s}^q + \|M^2(\tau)\|_{B_{p,r}^s}^q + 1) d\tau \\
&\quad + C \int_0^t \|\partial_x H^1\|_{B_{p,r}^s} \|M^{12}(\tau)\|_{B_{p,r}^{s-1}} d\tau \\
&\leq \|M_0^{12}\|_{B_{p,r}^{s-1}} + C \int_0^t \|M^{12}(\tau)\|_{B_{p,r}^{s-1}} (\|M^1(\tau)\|_{B_{p,r}^s}^q + \|M^2(\tau)\|_{B_{p,r}^s}^q + 1) d\tau \\
&\quad + C \int_0^t (\|U^1\|_{B_{p,r}^s}^l + \|U_x^1\|_{B_{p,r}^s}^l + \|U_{xx}^1\|_{B_{p,r}^s}^l) \|M^{12}(\tau)\|_{B_{p,r}^{s-1}} d\tau \\
&\leq \|M_0^{12}\|_{B_{p,r}^{s-1}} + C \int_0^t \|M^{12}(\tau)\|_{B_{p,r}^{s-1}} (\|M^1(\tau)\|_{B_{p,r}^s}^q + \|M^2(\tau)\|_{B_{p,r}^s}^q + 1) d\tau.
\end{aligned}$$

Hence, the Gronwall lemma gives the inequality (3.2).

For the critical case (2)  $s = 2 + \frac{1}{p}$ , let us choose  $s_1 \in (\max\{1 - \frac{1}{p}, \frac{1}{p}\} - 1, s - 1)$ ,  $s_2 \in (s - 1, s)$ . Then  $s - 1 = \theta s_1 + (1 - \theta) s_2$  with  $\theta = \frac{s_2 - (s-1)}{s_2 - s_1} \in (0, 1)$ . By using the interpolation inequality and the consequence



of the case (1), we get

$$\begin{aligned}
\|M^{12}(t)\|_{B_{p,r}^{s-1}} &\leq \|M^{12}(t)\|_{B_{p,r}^{s_1}}^\theta \|M^{12}(t)\|_{B_{p,r}^{s_2}}^{1-\theta} \\
&\leq (\|M_0^{12}\|_{B_{p,r}^{s_1}} e^{C \int_0^t (\|M^1(\tau)\|_{B_{p,r}^{s_1+1}}^q + \|M^2(\tau)\|_{B_{p,r}^{s_1+1}}^q) d\tau})^\theta (\|M^1(t)\|_{B_{p,r}^{s_2}} + \|M^2(t)\|_{B_{p,r}^{s_2}})^{1-\theta} \\
&\leq \|M_0^{12}\|_{B_{p,r}^{s-1}}^\theta (\|M^1(t)\|_{B_{p,r}^s} + \|M^2(t)\|_{B_{p,r}^s})^{1-\theta} e^{\theta C \int_0^t (\|M^1(\tau)\|_{B_{p,r}^s}^q + \|M^2(\tau)\|_{B_{p,r}^s}^q) d\tau},
\end{aligned}$$

which completes the proof of the lemma.  $\square$

Proof of Theorem 3.1. Since uniqueness in Theorem 3.1 is a straightforward corollary of Lemma 3.1, we need only to prove the existence of a solution to Eq.(3.1). We shall proceed as follows.

First step: constructing approximate solutions.

Starting from  $M^0 = M_0$  we define by induction a sequence  $(M^n)_{n \in \mathbb{N}}$  by solving the following linear transport equation

$$(3.6) \quad \begin{cases} M_t^{n+1} - H^n M_x^{n+1} = A^n M^n + B^n M^n, \\ M_{|t=0}^{n+1} = M_0, \end{cases}$$

where  $M^n = (m_1^n, \dots, m_N^n, n_1^n, \dots, n_N^n)^T$ ,  $U^n = (u_1^n, \dots, u_N^n, v_1^n, \dots, v_N^n)^T$ ,  $H^n = H(U^n, U_x^n)$ ,  $A^n = A(H^n, H_x^n)$  and  $B^n = B(U^n, U_x^n)$ .

Second step: uniform bounds.

Let  $q = \max\{l, 2\}$ . The condition  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$  yields that  $B_{p,r}^s$  is an algebra. Thus, we have

$$\begin{aligned}
\|A^n M^n + B^n M^n\|_{B_{p,r}^s} &\leq (\|A^n\|_{B_{p,r}^s} + \|B^n\|_{B_{p,r}^s}) \|M^n\|_{B_{p,r}^s} \\
&\lesssim (1 + \|U^n\|_{B_{p,r}^s}^l + \|U_x^n\|_{B_{p,r}^s}^l + \|U_{xx}^n\|_{B_{p,r}^s}^l + \|U^n\|_{B_{p,r}^s}^2 + \|U_x^n\|_{B_{p,r}^s}^2) \|M^n\|_{B_{p,r}^s} \\
&\lesssim (1 + \|M^n\|_{B_{p,r}^s}^q) \|M^n\|_{B_{p,r}^s}.
\end{aligned}$$

According to Lemma 2.5 with the above inequality and

$$\begin{cases} p_1 = p, & \text{if } \max\{1 - \frac{1}{p}, \frac{1}{p}\} < s, s \neq 1 + \frac{1}{p}, \\ p_1 = \infty, & \text{if } \max\{1 - \frac{1}{p}, \frac{1}{p}\} < s, s = 1 + \frac{1}{p}, (\text{which implies } p \neq \infty) \end{cases}$$

and

$$\begin{aligned}
V_{p_1}'(t) &= \begin{cases} \|\partial_x H^n\|_{B_{p,r}^{s-1}} + \|\partial_x H^n\|_{B_{p,r}^{\frac{1}{p}} \cap L^\infty}, & \text{if } \max\{1 - \frac{1}{p}, \frac{1}{p}\} < s, s \neq 1 + \frac{1}{p}, \\ \|\partial_x H^n\|_{B_{\infty,r}^{s-1}} \lesssim \|\partial_x H^n\|_{B_{p,r}^{s-1+\frac{1}{p}}}, & \text{if } \max\{1 - \frac{1}{p}, \frac{1}{p}\} < s, s = 1 + \frac{1}{p}, \end{cases} \\
&\lesssim \begin{cases} \|\partial_x H^n\|_{B_{p,r}^s}, & \text{if } \max\{1 - \frac{1}{p}, \frac{1}{p}\} < s, s \neq 1 + \frac{1}{p}, \\ \|\partial_x H^n\|_{B_{p,r}^s}, & \text{if } \max\{1 - \frac{1}{p}, \frac{1}{p}\} < s, s = 1 + \frac{1}{p}, \end{cases}
\end{aligned}$$

we get

$$\begin{aligned}
\|M^{n+1}(t)\|_{B_{p,r}^s} &\leq \|M_0\|_{B_{p,r}^s} + C \int_0^t (1 + \|M^n(\tau)\|_{B_{p,r}^s}^q) \|M^n(\tau)\|_{B_{p,r}^s} d\tau \\
&\quad + C \int_0^t \|\partial_x H^n(\tau)\|_{B_{p,r}^s} \|M^{n+1}(\tau)\|_{B_{p,r}^s} d\tau \\
&\leq \|M_0\|_{B_{p,r}^s} + C \int_0^t (1 + \|M^n(\tau)\|_{B_{p,r}^s}^q) \|M^n(\tau)\|_{B_{p,r}^s} d\tau \\
&\quad + C \int_0^t (\|U^n\|_{B_{p,r}^s}^l + \|U_x^n\|_{B_{p,r}^s}^l + \|U_{xx}^n\|_{B_{p,r}^s}^l) \|M^{n+1}(\tau)\|_{B_{p,r}^s} d\tau \\
&\leq \|M_0\|_{B_{p,r}^s} + C \int_0^t (1 + \|M^n(\tau)\|_{B_{p,r}^s}^q) \|M^n(\tau)\|_{B_{p,r}^s} d\tau \\
&\quad + C \int_0^t (\|M^n(\tau)\|_{B_{p,r}^s}^q + 1) \|M^{n+1}(\tau)\|_{B_{p,r}^s} d\tau.
\end{aligned}$$

The Gronwall lemma yields that

$$\begin{aligned}
(3.7) \quad \|M^{n+1}(t)\|_{B_{p,r}^s} &\leq e^{C \int_0^t (1 + \|M^n(\tau)\|_{B_{p,r}^s}^q) d\tau} \left( \|M_0\|_{B_{p,r}^s} \right. \\
&\quad \left. + C \int_0^t e^{-C \int_0^\tau (1 + \|M^n(t')\|_{B_{p,r}^s}^q) dt'} (1 + \|M^n(\tau)\|_{B_{p,r}^s}^q) \|M^n(\tau)\|_{B_{p,r}^s} d\tau \right).
\end{aligned}$$

Notice that  $f(t) = \frac{f_0 e^{2Ct}}{(1 + f_0^q - f_0^q e^{2Ctq})^{\frac{1}{q}}}$  is the solution to the following equation:

$$(3.8) \quad f(t) = e^{C \int_0^t (1 + f^q(\tau)) d\tau} \left( f_0 + C \int_0^t e^{-C \int_0^\tau (1 + f^q(t') dt') d\tau} (1 + f^q(\tau)) f(\tau) d\tau \right).$$

We fix a  $T > 0$  such that  $1 + \|M_0\|_{B_{p,r}^s}^q - \|M_0\|_{B_{p,r}^s}^q e^{2qCT} > 0$  and suppose that

$$\forall t \in [0, T], \quad \|M^n\|_{B_{p,r}^s} \leq \frac{\|M_0\|_{B_{p,r}^s} e^{2Ct}}{(1 + \|M_0\|_{B_{p,r}^s}^q - \|M_0\|_{B_{p,r}^s}^q e^{2Ctq})^{\frac{1}{q}}}.$$

Plugging the above inequality into (3.7) and using (3.8) yield

$$\|M^{n+1}(t)\|_{B_{p,r}^s} \leq \frac{\|M_0\|_{B_{p,r}^s} e^{2Ct}}{(1 + \|M_0\|_{B_{p,r}^s}^q - \|M_0\|_{B_{p,r}^s}^q e^{2Ctq})^{\frac{1}{q}}}.$$

Therefore,  $(M^n)_{n \in \mathbb{N}}$  is bounded in  $L^\infty(0, T; B_{p,r}^s)$ .

Third step: convergence.

Similar to the proof of (3.5), we have, for  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$  and  $s \neq 2 + \frac{1}{p}$ ,

$$\begin{aligned}
&\|(M^{n+m+1} - M^{n+1})(t)\|_{B_{p,r}^{s-1}} \\
&\leq C \int_0^t \|(M^{n+m} - M^n)(\tau)\|_{B_{p,r}^{s-1}} (\|M^n(\tau)\|_{B_{p,r}^s}^q + \|M^{n+1}(\tau)\|_{B_{p,r}^s}^q + \|M^{n+m}(\tau)\|_{B_{p,r}^s}^q + 1) d\tau \\
&\quad + C \int_0^t \|(M^{n+m+1} - M^{n+1})(\tau)\|_{B_{p,r}^{s-1}} (\|M^{n+m}(\tau)\|_{B_{p,r}^s}^q + 1) d\tau.
\end{aligned}$$

Taking advantage of the Gronwall inequality gives

$$\begin{aligned}
& \| (M^{n+m+1} - M^{n+1})(t) \|_{B_{p,r}^{s-1}} \\
& \leq C e^{C \int_0^t (\|M^{n+m}(t')\|_{B_{p,r}^s}^q + 1) dt'} \int_0^t e^{-C \int_0^\tau (\|M^{n+m}(t')\|_{B_{p,r}^s}^q + 1) dt'} \| (M^{n+m} - M^n)(\tau) \|_{B_{p,r}^{s-1}} \\
& \quad \times (\|M^n(\tau)\|_{B_{p,r}^s}^q + \|M^{n+1}(\tau)\|_{B_{p,r}^s}^q + \|M^{n+m}(\tau)\|_{B_{p,r}^s}^q + 1) d\tau.
\end{aligned}$$

Since  $(M^n)_{n \in \mathbb{N}}$  is bounded in  $L^\infty(0, T; B_{p,r}^s)$ , we finally get a constant  $C_T$ , independent of  $n$  and  $m$ , such that

$$\| (M^{n+m+1} - M^{n+1})(t) \|_{B_{p,r}^{s-1}} \leq C_T \int_0^t \| (M^{n+m} - M^n)(\tau) \|_{B_{p,r}^{s-1}} d\tau.$$

Finally, arguing by induction, we arrive at

$$\| (M^{n+m+1} - M^{n+1})(t) \|_{B_{p,r}^{s-1}} \leq \frac{(TC_T)^{n+1}}{(n+1)!} \|M^m - M^0\|_{L_T^\infty(B_{p,r}^{s-1})} \leq C_T \frac{(TC_T)^{n+1}}{(n+1)!},$$

which implies that  $(M^n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^\infty(0, T; B_{p,r}^{s-1})$ .

For the critical case  $s = 2 + \frac{1}{p}$ , from the above argument, we get that  $(M^n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^\infty(0, T; B_{p,r}^{s-1-\varepsilon})$  with sufficiently small  $\varepsilon$ . Then applying the interpolation method with uniform bounds in  $L^\infty(0, T; B_{p,r}^s)$  obtained in the second step, we show that  $(M^n)_{n \in \mathbb{N}}$  is also a Cauchy sequence in  $L^\infty(0, T; B_{p,r}^{s-1})$  for the critical case.

Final step: conclusion.

Let  $M$  be the limit of the sequence  $(M^n)_{n \in \mathbb{N}}$  in  $L^\infty(0, T; B_{p,r}^{s-1})$ . According to the Fatou lemma 2.2,  $M$  also belongs to  $L^\infty(0, T; B_{p,r}^s)$ . It is then easy to pass to the limit in Eq.(3.6) and to conclude that  $M$  is a solution of Eq.(3.1). Note that  $A(U, U_x)M + B(H, H_x)M$  of Eq.(3.1) also belongs to  $L^\infty(0, T; B_{p,r}^s)$ . According to Lemma 2.5, we have  $M \in C([0, T]; B_{p,r}^s)$  if  $r < \infty$ , or  $M \in \left( \bigcap_{s' < s} C([0, T]; B_{p,r}^{s'}) \right)$ , if  $r = \infty$ . Again using the equation, we see that  $M_t \in C([0, T]; B_{p,r}^{s-1})$  if  $r < \infty$ , or  $M_t \in \left( \bigcap_{s' < s} C([0, T]; B_{p,r}^{s'-1}) \right)$ , if  $r = \infty$ . This completes the proof of Theorem 3.1.  $\square$

### 3.2. A continuation criterion

In this subsection, we state a continuation criterion for Eq.(3.1).

**Theorem 3.2.** *Let  $M_0 \in B_{p,r}^s$  with  $1 \leq p, r \leq \infty$ ,  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$  and  $T > 0$  be the maximal existence time of the corresponding solution  $M$  to Eq.(3.1). If  $T$  is finite, then we have*

$$\int_0^T \|M(\tau)\|_{L^\infty}^q d\tau = \infty,$$

where  $q = \max\{l, 2\}$  ( $l$  is the polynomial order of  $H$ ).

Proof. For any  $0 < \sigma \leq s$ , applying Proposition 2.4 (i), we have

$$\begin{aligned}
(3.9) \quad & \|A(H, H_x)M + B(U, U_x)M\|_{B_{p,r}^\sigma} \\
& \leq (\|A(H, H_x)\|_{L^\infty} + \|B(U, U_x)\|_{L^\infty})\|M\|_{B_{p,r}^\sigma} + (\|A(H, H_x)\|_{B_{p,r}^\sigma} + \|B(U, U_x)\|_{B_{p,r}^\sigma})\|M\|_{L^\infty} \\
& \leq (\|U\|_{L^\infty}^l + \|U_x\|_{L^\infty}^l + \|U_{xx}\|_{L^\infty}^l + 1 + \|U\|_{L^\infty}^2 + \|U_x\|_{L^\infty}^2)\|M\|_{B_{p,r}^\sigma} \\
& \quad + \left( (\|U\|_{B_{p,r}^\sigma} + \|U_x\|_{B_{p,r}^\sigma} + \|U_{xx}\|_{B_{p,r}^\sigma} + 1)(\|U\|_{L^\infty}^{l-1} + \|U_x\|_{L^\infty}^{l-1} + \|U_{xx}\|_{L^\infty}^{l-1} + 1) \right. \\
& \quad \left. + (\|U\|_{B_{p,r}^\sigma} + \|U_x\|_{B_{p,r}^\sigma})(\|U\|_{L^\infty} + \|U_x\|_{L^\infty}) \right)\|M\|_{L^\infty} \\
& \leq (\|M\|_{L^\infty}^q + 1)(\|M\|_{B_{p,r}^\sigma} + 1).
\end{aligned}$$

We now consider the case  $1 < p < \infty$ .

Step 1. If  $\sigma > 1$ , then we claim that

$$(3.10) \quad \|M\|_{L_T^\infty(B_{p,r}^1)} < \infty, \text{ and } \int_0^T \|M(\tau)\|_{L^\infty}^q d\tau < \infty \Rightarrow \|M\|_{L_T^\infty(B_{p,r}^\sigma)} < \infty.$$

In fact, by using (3.9) and Lemma 2.5 with  $p_1 = \infty$  and

$$\begin{aligned}
V'_{p_1}(t) &= \|\partial_x H\|_{B_{\infty,r}^{\sigma-1}} \leq \|\partial_x H\|_{B_{p,r}^{\sigma-1+\frac{1}{p}}} \\
&\leq \|U\|_{B_{p,r}^{\sigma-1+\frac{1}{p}}}^l + \|U_x\|_{B_{p,r}^{\sigma-1+\frac{1}{p}}}^l + \|U_{xx}\|_{B_{p,r}^{\sigma-1+\frac{1}{p}}}^l \leq \|M\|_{B_{p,r}^{\sigma-1+\frac{1}{p}}}^l,
\end{aligned}$$

we have

$$\begin{aligned}
\|M(t)\|_{B_{p,r}^\sigma} &\leq \|M_0\|_{B_{p,r}^\sigma} + C \int_0^t (\|M(\tau)\|_{L^\infty}^q + 1)(\|M(\tau)\|_{B_{p,r}^\sigma} + 1) d\tau \\
&\quad + C \int_0^t \|M\|_{B_{p,r}^{\sigma-1+\frac{1}{p}}}^l \|M(\tau)\|_{B_{p,r}^\sigma} d\tau.
\end{aligned}$$

Hence, the Gronwall lemma gives

$$\|M(t)\|_{B_{p,r}^\sigma} + 1 \leq (\|M_0\|_{B_{p,r}^\sigma} + 1)e^{C \int_0^t (\|M(\tau)\|_{L^\infty}^q + 1 + \|M\|_{B_{p,r}^{\sigma-1+\frac{1}{p}}}^l) d\tau},$$

which implies

$$\|M\|_{L_T^\infty(B_{p,r}^{\sigma-1+\frac{1}{p}})} < \infty, \text{ and } \int_0^T \|M(\tau)\|_{L^\infty}^q d\tau < \infty \Rightarrow \|M\|_{L_T^\infty(B_{p,r}^\sigma)} < \infty.$$

If  $\sigma - 1 + \frac{1}{p} > 1$ , then repeat the above process. Clearly, this process stops within a finite number of steps.

Our claim (3.10) is guaranteed.

Step 2. If  $\sigma = 1$ , then by using (3.9) and Lemma 2.5 with  $p_1 = p$  and

$$V'_{p_1}(t) = \|\partial_x H\|_{B_{p,\infty}^{\frac{1}{p}} \cap L^\infty} \leq \|U\|_{B_{p,\infty}^{\frac{1}{p}} \cap L^\infty}^l + \|U_x\|_{B_{p,\infty}^{\frac{1}{p}} \cap L^\infty}^l + \|U_{xx}\|_{B_{p,\infty}^{\frac{1}{p}} \cap L^\infty}^l$$

$$\leq \|M\|_{B_{p,\infty}^{\frac{1}{p}} \cap L^\infty}^l \lesssim \|M\|_{B_{p,\infty}^{\frac{1}{p}}}^l + \|M\|_{L^\infty}^q + 1,$$

we have

$$\begin{aligned} \|M(t)\|_{B_{p,r}^\sigma} &\leq \|M_0\|_{B_{p,r}^\sigma} + C \int_0^t (\|M(\tau)\|_{L^\infty}^q + 1)(\|M(\tau)\|_{B_{p,r}^\sigma} + 1) d\tau \\ &\quad + C \int_0^t (\|M\|_{B_{p,r}^{\frac{1}{p}}}^l + \|M\|_{L^\infty}^q + 1) \|M(\tau)\|_{B_{p,r}^\sigma} d\tau. \end{aligned}$$

Hence, the Gronwall lemma gives

$$\|M(t)\|_{B_{p,r}^\sigma} + 1 \leq (\|M_0\|_{B_{p,r}^\sigma} + 1) e^{C \int_0^t (\|M(\tau)\|_{L^\infty}^q + 1 + \|M\|_{B_{p,r}^{\frac{1}{p}}}^l) d\tau},$$

which implies

$$(3.11) \quad \|M\|_{L_T^\infty(B_{p,r}^{\frac{1}{p}})} < \infty \text{ and } \int_0^T \|M(\tau)\|_{L^\infty}^q d\tau < \infty \Rightarrow \|M\|_{L_T^\infty(B_{p,r}^\sigma)} < \infty.$$

Step 3. If  $\sigma \in (0, 1)$ , applying Lemma 2.5 with  $p_1 = \infty$  and

$$\begin{aligned} V'_{p_1}(t) &= \|\partial_x H\|_{B_{\infty,\infty}^0 \cap L^\infty} \leq \|\partial_x H\|_{L^\infty} \\ &\leq \|U\|_{L^\infty}^l + \|U_x\|_{L^\infty}^l + \|U_{xx}\|_{L^\infty}^l \leq \|M\|_{L^\infty}^q + 1, \end{aligned}$$

we have

$$\begin{aligned} \|M(t)\|_{B_{p,r}^\sigma} &\leq \|M_0\|_{B_{p,r}^\sigma} + C \int_0^t (\|M(\tau)\|_{L^\infty}^q + 1)(\|M(\tau)\|_{B_{p,r}^\sigma} + 1) d\tau \\ &\quad + C \int_0^t (\|M\|_{L^\infty}^q + 1) \|M(\tau)\|_{B_{p,r}^\sigma} d\tau. \end{aligned}$$

Hence, the Gronwall lemma gives

$$\|M(t)\|_{B_{p,r}^\sigma} + 1 \leq (\|M_0\|_{B_{p,r}^\sigma} + 1) e^{C \int_0^t (\|M(\tau)\|_{L^\infty}^q + 1) d\tau},$$

which implies

$$(3.12) \quad \int_0^T \|M(\tau)\|_{L^\infty}^q d\tau < \infty \Rightarrow \|M\|_{L_T^\infty(B_{p,r}^\sigma)} < \infty.$$

Therefore, for all  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$ , if  $T < \infty$ , and  $\int_0^T \|M(\tau)\|_{L^\infty}^q d\tau < \infty$ , then we have  $\limsup_{t \rightarrow T} \|M(t)\|_{B_{p,r}^s} < \infty$ .

The cases  $p = 1$  and  $p = \infty$  can be treated similarly. We also have for  $s > 1$ , if  $T < \infty$ , and  $\int_0^T \|M(\tau)\|_{L^\infty}^q d\tau < \infty$ , then  $\limsup_{t \rightarrow T} \|M(t)\|_{B_{p,r}^s} < \infty$ . For the sake of simplicity, we omit the details here.<sup>1</sup>

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<sup>1</sup>We present a simple flow chart. For  $p = \infty$ ,  $\sigma > 1$ , let  $\varepsilon \in (0, 1)$ .  $\|M\|_{B_{\infty,r}^\sigma} \xrightarrow{\text{determined by}} \|\partial_x H\|_{B_{\infty,r}^{\sigma-1}} \lesssim \|M\|_{B_{\infty,r}^{\sigma-1}} \xrightarrow{\text{determined by}} \|M\|_{B_{\infty,r}^{\varepsilon+1}} \xrightarrow{\text{determined by}} \|\partial_x H\|_{B_{\infty,r}^\varepsilon} \lesssim \|M\|_{B_{\infty,r}^\varepsilon} \xrightarrow{\text{determined by}} \|\partial_x H\|_{B_{\infty,\infty}^0 \cap L^\infty} \lesssim \|M\|_{L^\infty}$ . For  $p = 1$ ,  $\sigma > 1$ , choose  $p_1$  such that  $1 < p_1 < \infty$  and  $\sigma > 1 + \frac{1}{p_1}$ .  $\|M\|_{B_{1,r}^\sigma} \xrightarrow{\text{determined by}} \|\partial_x H\|_{B_{p_1,r}^{\sigma-1}} \lesssim \|M\|_{B_{p_1,r}^{\sigma-1}} \xrightarrow{\text{determined by}} \text{turn to the case (1) } 1 < p_1 < \infty \xrightarrow{\text{determined by}} \|M\|_{L^\infty}$ .

Finally, if  $\limsup_{t \rightarrow T} \|M(t)\|_{B_{p,r}^s} < \infty$ , then by Theorem 3.1, we can extend the solution  $M$  beyond  $T$ , which is a contradiction with the assumption of  $T$ . Then we must have  $\int_0^T \|M(\tau)\|_{L^\infty}^q d\tau = \infty$ . This completes the proof of the theorem.  $\square$

Combining Theorem 3.1 and Theorem 3.2, we readily obtain the following corollary.

**Corollary 3.1.** *Let  $M_0 \in B_{p,r}^s$  with  $1 \leq p, r \leq \infty$  and  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}\}$  and  $T > 0$  be the maximal existence time of the corresponding solution  $M$  to Eq.(3.1). Then the solution  $M$  blows up in finite time if and only if  $\limsup_{t \rightarrow T} \|M(t)\|_{L^\infty} = \infty$ .*

**Remark 3.1.** *Apparently, for every  $s \in \mathbb{R}$ ,  $B_{2,2}^s = H^s$ . Theorem 3.1, Theorem 3.2 and Corollary 3.1 hold true in the corresponding Sobolev spaces  $H^s$  with  $s > \frac{1}{2}$ , which recovers the corresponding results in [36] and [25] as  $N = 1$ ,  $H = -\frac{1}{2}(u_1 v_1 - u_{1x} v_{1x})$  and  $N = 1$ ,  $H = -\frac{1}{2}(u_1 v_1 - u_{1x} v_{1x}), v = 2u$ , respectively.*

**Remark 3.2.** *We pointed out that if  $N = 1$  and  $H = -\frac{1}{2}(u_1 v_1 - u_{1x} v_{1x})$ , then Theorem 3.1 improves the corresponding result in [36], where  $s > \max\{1 - \frac{1}{p}, \frac{1}{p}, \frac{1}{2}\}$  but  $s \neq 1 + \frac{1}{p}$ . Besides, Theorem 3.2, which works in Besov spaces, also improves the corresponding result in [36], where the corresponding blow-up scenario works only in Sobolev spaces.*

## 4 Global existence and blow-up phenomena for the two-component subsystems

4.1.  $N = 1, H = -\frac{1}{2}(u - u_x)(v + v_x)$

### 4.1.1 A precise blow-up scenario

As mentioned in the Introduction, for  $N = 1$  and  $H = -\frac{1}{2}(u - u_x)(v + v_x)$ , Eq.(1.1) is reduced to the following system:

$$(4.1) \quad \begin{cases} m_t + \frac{1}{2}((u - u_x)(v + v_x)m)_x = 0, \\ n_t + \frac{1}{2}((u - u_x)(v + v_x)n)_x = 0, \\ (m, n)|_{t=0} = (m_0, n_0), \end{cases}$$

where  $m = u - u_{xx}$  and  $n = v - v_{xx}$ .

Consider the following initial value problem

$$(4.2) \quad \begin{cases} q_t(t, x) = \frac{1}{2}(u - u_x)(v + v_x)(t, q), & t \in [0, T), \\ q(0, x) = x, & x \in \mathbb{R}. \end{cases}$$

---

**Lemma 4.1.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.1). Then Eq.(4.2) has a unique solution  $q \in C^1([0, T] \times \mathbb{R}; \mathbb{R})$ . Moreover, the mapping  $q(t, \cdot)$  ( $t \in [0, T]$ ) is an increasing diffeomorphism of  $\mathbb{R}$ , with*

$$(4.3) \quad q_x(t, x) = \exp\left(\int_0^t \frac{1}{2}(m(v + v_x) - n(u - u_x))(\tau, q(\tau, x))d\tau\right).$$

Proof. According to Remark 3.1, we get that  $m, n \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$  with  $s > \frac{1}{2}$ , from which we deduce that  $\frac{1}{2}(u - u_x)(v + v_x)$  is bounded and Lipschitz continuous in the space variable  $x$  and of class  $C^1$  in time variable  $t$ , then the classical ODE theory ensures that Eq.(4.2) has a unique solution  $q \in C^1([0, T] \times \mathbb{R}; \mathbb{R})$ . Differentiating Eq.(4.2) with respect to  $x$  gives

$$(4.4) \quad \begin{cases} q_{xt}(t, x) = \frac{1}{2}(m(v + v_x) - n(u - u_x))(t, q)q_x(t, x), & t \in [0, T], \\ q_x(0, x) = 1, & x \in \mathbb{R}, \end{cases}$$

which leads to (4.3). So, the mapping  $q(t, \cdot)$  ( $t \in [0, T]$ ) is an increasing diffeomorphism of  $\mathbb{R}$ .  $\square$

**Lemma 4.2.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.1). Then, we have for all  $t \in [0, T]$ ,*

$$(4.5) \quad m(t, q(t, x))q_x(t, x) = m_0(x),$$

$$(4.6) \quad n(t, q(t, x))q_x(t, x) = n_0(x).$$

Proof. Combining Eq.(4.2), Lemma 4.1 and Eq.(4.1), we have

$$\begin{aligned} \frac{d}{dt}(m(t, q(t, x))q_x(t, x)) &= (m_t(t, q) + m_x(t, q)q_t(t, x))q_x(t, x) + m(t, q)q_{xt}(t, x) \\ &= (m_t(t, q) + \frac{1}{2}((u - u_x)(v + v_x)m)_x(t, q))q_x(t, x) = 0. \end{aligned}$$

Therefore, the Gronwall inequality yields (4.5). Similar arguments lead to (4.6). This completes the proof of the lemma.  $\square$

The following theorem shows a precise blow-up scenario for Eq.(4.1).

**Theorem 4.1.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.1). Then the solution  $(m, n)$  blows up in finite time if and only if*

$$\liminf_{t \rightarrow T} \inf_{x \in \mathbb{R}} (m(v + v_x) - n(u - u_x))(t, x) = -\infty.$$

Proof. Assume that the solution  $(m, n)$  blows up in finite time  $T$  and there exists a constant  $C$  such that

$$(m(v + v_x) - n(u - u_x))(t, x) \geq -C, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

By (4.3) and Lemma 4.2, we have that

$$\|m(t)\|_{L^\infty} + \|n(t)\|_{L^\infty} \leq (\|m_0\|_{L^\infty} + \|n_0\|_{L^\infty})e^{Ct}, \quad \forall t \in [0, T),$$

which contradicts to Corollary 3.1.

On the other hand, if  $\liminf_{t \rightarrow T} \inf_{x \in \mathbb{R}} (m(v + v_x) - n(u - u_x))(t, x) = -\infty$ , then we can get

$$\limsup_{t \rightarrow T} \|m(t)\|_{L^\infty} = \infty \text{ or } \limsup_{t \rightarrow T} \|n(t)\|_{L^\infty} = \infty.$$

Thus according to Corollary 3.1, the solution  $(m, n)$  blows up. This completes the proof of the theorem.  $\square$

#### 4.1.2 Global existence

We now give a global existence result.

**Theorem 4.2.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ). Assume that  $\text{supp } m_0 \in [b, \infty)$ ,  $\text{supp } n_0 \in (-\infty, a]$ , with  $a \leq b$ . Then the corresponding solution  $(m, n)$  to Eq.(4.1) exists globally in time.*

Proof. Note that, according to Lemma 4.1, the function  $q(t, x)$  is an increasing diffeomorphism of  $\mathbb{R}$  with  $q_x(t, x) > 0$  with respect to time  $t$ . Thus  $a \leq b$  implies  $q(t, a) \leq q(t, b)$ . We infer from Lemma 4.1 and Lemma 4.2 that for all  $t \in [0, T)$ , we have

$$(4.7) \quad \begin{cases} m(t, x) = 0, & \text{if } x < q(t, b), \\ n(t, x) = 0, & \text{if } x > q(t, a). \end{cases}$$

Noticing

$$\begin{aligned} u(t, x) - u_x(t, x) &= e^{-x} \int_{-\infty}^x e^y m(t, y) dy, \\ v(t, x) + v_x(t, x) &= e^x \int_x^{\infty} e^{-y} n(t, y) dy, \end{aligned}$$

we have

$$(4.8) \quad \begin{cases} u(t, x) - u_x(t, x) = 0, & \text{if } x \leq q(t, b), \\ v(t, x) + v_x(t, x) = 0, & \text{if } x \geq q(t, a). \end{cases}$$

Therefore, for Eq.(4.1),  $(m(v + v_x) - n(u - u_x))(t, x) = 0$  on  $\mathbb{R}$  for all  $t \in [0, T)$ . Then Theorem 4.1 implies  $T = \infty$ . This proves the solution  $(m, n)$  exists globally in time.  $\square$

#### 4.1.3 Blow-up phenomena

As a straight corollary of Lemma 4.1- 4.2, we have the following lemma.



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**Lemma 4.3.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.1). Assume further  $m_0, n_0 \in L^1$ . Then we have for all  $t \in [0, T)$ ,*

$$\|m(t)\|_{L^1} = \|m_0\|_{L^1}, \|n(t)\|_{L^1} = \|n_0\|_{L^1}.$$

Now we derive two useful conservation laws for Eq.(4.1).

**Lemma 4.4.** *Let  $m_0, n_0 \in H^s$  with  $s > \frac{1}{2}$ , and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.1). Then we have that for all  $t \in [0, T)$ ,*

$$\begin{aligned} \int_{\mathbb{R}} m(v + v_x)(t, x) dx &= \int_{\mathbb{R}} m_0(v_0 + v_{0x}) dx, \\ \int_{\mathbb{R}} n(u - u_x)(t, x) dx &= \int_{\mathbb{R}} n_0(u_0 - u_{0x}) dx. \end{aligned}$$

Proof. By Eq.(4.1), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} m(v + v_x)(t, x) dx &= \frac{d}{dt} \int_{\mathbb{R}} n(u - u_x)(t, x) dx \\ &= \int_{\mathbb{R}} ((v + v_x)m_t + (u - u_x)n_t)(t, x) dx \\ &= \frac{1}{2} \int_{\mathbb{R}} (u - u_x)(v + v_x)(m(v_x + v_{xx}) + n(u_x - u_{xx}))(t, x) dx \\ &= \frac{1}{2} \int_{\mathbb{R}} (u - u_x)(v + v_x)(m(v + v_x) - n(u - u_x))(t, x) dx \\ &= \frac{1}{2} \int_{\mathbb{R}} (u - u_x)(v + v_x) \partial_x ((u - u_x)(v + v_x))(t, x) dx = 0. \end{aligned}$$

This completes the proof of the lemma.  $\square$

**Lemma 4.5.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.1). Assume that  $m_0$  and  $n_0$  do not change sign. Then there exists a constant  $C = C(\|(v_0 + v_{0x})m_0\|_{L^1}, \|(u_0 - u_{0x})n_0\|_{L^1}, \|u_0\|_{H^1}, \|v_0\|_{H^1})$  such that*

$$(4.9) \quad |u_x(t, x)| \leq |u(t, x)|, \quad |v_x(t, x)| \leq |v(t, x)|,$$

$$(4.10) \quad \|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq Ce^{Ct}, \quad \forall t \in [0, T).$$

Proof. One can assume without loss of generality that  $m_0 \geq 0$ ,  $n_0 \geq 0$  for all  $x \in \mathbb{R}$ . Since  $m_0 \geq 0$ , (4.3) and (4.5) imply that

$$(4.11) \quad m(t, x) \geq 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Noticing

$$u(t, x) = (1 - \partial_x^2)^{-1} m(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} m(t, y) dy,$$

we obtain

$$u(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^x e^y m(t, y) dy + \frac{e^x}{2} \int_x^{\infty} e^{-y} m(t, y) dy,$$

and

$$u_x(t, x) = -\frac{e^{-x}}{2} \int_{-\infty}^x e^y m(t, y) dy + \frac{e^x}{2} \int_x^{\infty} e^{-y} m(t, y) dy,$$

which lead to

$$(4.12) \quad u(t, x) + u_x(t, x) = e^x \int_x^{\infty} e^{-y} m(t, y) dy \geq 0,$$

$$(4.13) \quad u(t, x) - u_x(t, x) = e^{-x} \int_{-\infty}^x e^y m(t, y) dy \geq 0.$$

From the above two inequalities, we have

$$(4.14) \quad |u_x(t, x)| \leq u(t, x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Similar arguments lead to

$$(4.15) \quad n(t, x) \geq 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

$$(4.16) \quad v(t, x) + v_x(t, x) = e^x \int_{-\infty}^x e^{-y} n(t, y) dy \geq 0,$$

$$(4.17) \quad v(t, x) - v_x(t, x) = e^{-x} \int_{-\infty}^x e^y n(t, y) dy \geq 0,$$

$$(4.18) \quad |v_x(t, x)| \leq v(t, x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Using Eq.(4.1), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2) = \int_{\mathbb{R}} (m_t u + n_t v)(t, x) dx \\ &= \frac{1}{2} \int_{\mathbb{R}} ((u - u_x)(v + v_x) m u_x + (u - u_x)(v + v_x) n v_x)(t, x) dx \\ &\leq \frac{1}{2} (\|(u - u_x) u_x(t)\|_{L^\infty} \|((v + v_x) m)(t)\|_{L^1} + \|((v + v_x) v_x)(t)\|_{L^\infty} \|((u - u_x) n)(t)\|_{L^1}). \end{aligned}$$

Using (4.14) and (4.18), it yields that

$$\begin{aligned} & \|((u - u_x) u_x)(t)\|_{L^\infty} \leq 2 \|u(t)\|_{L^\infty}^2 \leq \|u(t)\|_{H^1}^2, \\ & \|((v + v_x) v_x)(t)\|_{L^\infty} \leq 2 \|v(t)\|_{L^\infty}^2 \leq \|v(t)\|_{H^1}^2. \end{aligned}$$

Using Lemma 4.4 with the fact that  $m, n, u - u_x, v + v_x \geq 0$ , we obtain

$$\|((v + v_x) m)(t)\|_{L^1} = \|(v_0 + v_{0x}) m_0\|_{L^1},$$

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$$\|((u - u_x)n)(t)\|_{L^1} = \|(u_0 - u_{0x})n_0\|_{L^1},$$

Combining the above three relations, we deduce that

$$\frac{d}{dt}(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2) \leq \frac{1}{2}(\|(v_0 + v_{0x})m_0\|_{L^1} + \|(u_0 - u_{0x})n_0\|_{L^1})(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2).$$

Gronwall's inequality then yields the desired inequality (4.10). This completes the proof of the lemma.  $\square$

**Lemma 4.6.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.1). Assume further  $m_0, n_0 \in L^1$ . Set  $Q(t, x) = \frac{1}{2}(u - u_x)(v + v_x)(t, x)$ . Then there exists a constant  $C = C(\|m_0\|_{L^1}, \|n_0\|_{L^1})$  such that for all  $t \in [0, T)$ ,*

$$(4.19) \quad Q_{xt}(t, x) + (Q(Q_x)_x)(t, x) + Q_x^2(t, x) \leq C(|m| + |n|)(t, x).$$

Proof. It is easy to deduce from Eq.(4.1) that

$$(4.20) \quad \begin{aligned} & Q_{xt}(t, x) + (Q(Q_x)_x)(t, x) + Q_x^2(t, x) \\ &= [-(1 - \partial_x^2)^{-1}((Q_x v) + \partial_x(Q_x v_x))m - (1 - \partial_x^2)^{-1}(\partial_x(Q_x v) + (Q_x v_x))m \\ &\quad + (1 - \partial_x^2)^{-1}((Q_x u) + \partial_x(Q_x u_x))n - (1 - \partial_x^2)^{-1}(\partial_x(Q_x u) + (Q_x u_x))n](t, x), \end{aligned}$$

where  $Q_x = \frac{1}{2}(m(v + v_x) - n(u - u_x))$ . Applying Lemma 4.3, we arrive at

$$\begin{aligned} & (1 - \partial_x^2)^{-1}((Q_x v) + \partial_x(Q_x v_x))(t, x)m(t, x) \\ & \leq \|(1 - \partial_x^2)^{-1}((Q_x v) + \partial_x(Q_x v_x))(t)\|_{L^\infty} |m(t, x)| \\ & \leq \frac{1}{2}e^{-|x|} \|Q_x(t)v(t)\|_{L^1} + \|Q_x(t)v_x(t)\|_{L^1} |m(t, x)| \\ & \leq C\|Q_x(t)\|_{L^1} (\|v(t)\|_{L^\infty} + \|v_x(t)\|_{L^\infty}) |m(t, x)| \\ & \leq C(\|m(t)\|_{L^1} + \|n(t)\|_{L^1}) (\|(u(t) - u_x(t))\|_{L^\infty} + \|(v(t) + v_x(t))\|_{L^\infty}) (\|v(t)\|_{L^\infty} + \|v_x(t)\|_{L^\infty}) |m(t, x)| \\ & \leq C(\|m(t)\|_{L^1} + \|n(t)\|_{L^1}) \|e^{-|x|}\|_{L^\infty} (\|m(t)\|_{L^1} + \|n(t)\|_{L^1}) \|e^{-|x|}\|_{L^\infty} (\|m(t)\|_{L^1} + \|n(t)\|_{L^1}) |m(t, x)| \\ & \leq C|m(t, x)|. \end{aligned}$$

Following along almost the same lines as above yields

$$\begin{aligned} & \|(1 - \partial_x^2)^{-1}(\partial_x(Q_x v) + (Q_x v_x))(t, x)m(t, x)\|_{L^\infty} \leq C|m(t, x)|, \\ & \|(1 - \partial_x^2)^{-1}((Q_x u) - \partial_x(Q_x u_x))(t, x)n(t, x) - (1 - \partial_x^2)^{-1}(\partial_x(Q_x u) - (Q_x u_x))(t, x)n(t, x)\|_{L^\infty} \leq C|n(t, x)|. \end{aligned}$$

Combining the above there inequalities completes the proof of the lemma.  $\square$

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**Lemma 4.7.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.1). Assume that  $m_0$  and  $n_0$  do not change sign. Set  $Q(t, x) = \frac{1}{2}(u - u_x)(v + v_x)(t, x)$ . Then there exists a constant  $C = C(\|(v_0 + v_{0x})m_0\|_{L^1}, \|(u_0 - u_{0x})n_0\|_{L^1}, \|u_0\|_{H^1}, \|v_0\|_{H^1})$  such that*

$$(4.21) \quad Q_{xt}(t, x) + (Q(Q_x)_x)(t, x) + Q_x^2(t, x) \leq Ce^{Ct}(|m| + |n|)(t, x).$$

Proof. Applying Lemma 4.5 to the first term on the right hand side of (4.20) yields

$$\begin{aligned} & (1 - \partial_x^2)^{-1}((Q_x v) + \partial_x(Q_x v_x))(t, x)m(t, x) \\ & \leq \|(1 - \partial_x^2)^{-1}((Q_x v) + \partial_x(Q_x v_x))(t)\|_{L^\infty} |m(t, x)| \\ & = \left\| \frac{1}{2} e^{-|x|} * \left( (m(v + v_x) - n(u - u_x))v \right) \right\|_{L^\infty} \\ & \quad + \left\| \frac{1}{2} (\text{sign}(x) e^{-|x|}) * \left( (m(v + v_x) - n(u - u_x))v_x \right) \right\|_{L^\infty} |m(t, x)| \\ & \leq C(\|u - u_x\|_{L^\infty} + \|v + v_x\|_{L^\infty})(\|v\|_{L^\infty} + \|v_x\|_{L^\infty})(\|e^{-|x|} * m\|_{L^\infty} + \|e^{-|x|} * n\|_{L^\infty})|m(t, x)| \\ & = C(\|u - u_x\|_{L^\infty} + \|v + v_x\|_{L^\infty})(\|v\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{L^\infty} + \|v\|_{L^\infty})|m(t, x)| \\ & \leq Ce^{Ct}|m|(t, x), \end{aligned}$$

where we have used the fact that  $m, n$  do not change sign. The left three terms can be treated in the same way. We have

$$\begin{aligned} & -(1 - \partial_x^2)^{-1}(\partial_x(Q_x v) + (Q_x v_x))m + (1 - \partial_x^2)^{-1}((Q_x u) - \partial_x(Q_x u_x))n \\ & - (1 - \partial_x^2)^{-1}(\partial_x(Q_x u) - (Q_x u_x))n(t, x) \leq Ce^{Ct}(|m| + |n|)(t, x). \end{aligned}$$

Plunging the above two inequalities into (4.20) completes the proof of the lemma.  $\square$

Next, we present two blow-up results.

**Theorem 4.3.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.1). Set  $Q(t, x) = \frac{1}{2}(u - u_x)(v + v_x)(t, x)$ . Assume that  $m_0$  and  $n_0$  do not change sign, and that there exists some  $x_0 \in \mathbb{R}$  such that  $N(0, x_0) = |m(0, x_0)| + |n(0, x_0)| > 0$  and  $Q_x(0, x_0) = \frac{1}{2}(m_0(v_0 + v_{0x}) - n_0(u_0 - u_{0x}))(x_0) \leq a_0$ , where  $a_0$  is the unique negative solution to the following equation*

$$1 + ag\left(-\frac{a}{N(0, x_0)}\right) + N(0, x_0) \int_0^{g\left(-\frac{a}{N(0, x_0)}\right)} f(s) ds = 0,$$

with  $f(x) = e^{Cx} - 1$ ,  $x \geq 0$ ,  $g(x) = \frac{1}{C} \log(x + 1)$ ,  $x \geq 0$ .

Then the solution  $(m, n)$  blows up at a time  $T_0 \leq g\left(-\frac{Q_x(0, x_0)}{N(0, x_0)}\right)$ .

Proof. In view of Lemma 4.6, we obtain that

$$Q_{xt}(t, x_0) + (Q(Q_x)_x)(t, x_0) + Q_x^2(t, x_0) \leq Ce^{Ct}(|m| + |n|)(t, x_0).$$

By Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned} \frac{d}{dt}Q_x(t, q(t, x_0)) + Q_x^2(t, q(t, x_0)) &\leq Ce^{Ct}(|m| + |n|)(t, q(t, x_0)) = Ce^{Ct}(|m_0(x_0)| + |n_0(x_0)|)q_x^{-1}(t, x_0) \\ &= CN(0, x_0)e^{Ct}\exp\left(\int_0^t -Q_x(\tau, q(\tau, x_0))d\tau\right), \end{aligned}$$

form which it follows that

$$\frac{d}{dt}(Q_x(t, q(t, x_0))\exp\left(\int_0^t Q_x(\tau, q(\tau, x_0))d\tau\right)) \leq CN(0, x_0)e^{Ct}.$$

Integrating from 0 to  $t$  yields

$$\frac{d}{dt}\exp\left(\int_0^t Q_x(\tau, q(\tau, x_0))d\tau\right) = Q_x(t, q(t, x_0))\exp\left(\int_0^t Q_x(\tau, q(\tau, x_0))d\tau\right) \leq N(0, x_0)(e^{Ct} - 1) + Q_x(0, x_0).$$

Integrating again from 0 to  $t$  yields

$$(4.22) \quad (e^{\int_0^t \inf_{x \in \mathbb{R}} Q_x(\tau, x)d\tau} \leq) \exp\left(\int_0^t Q_x(\tau, q(\tau, x_0))d\tau\right) \leq N(0, x_0) \int_0^t (e^{Cs} - 1)ds + Q_x(0, x_0)t + 1.$$

Next, we consider the following function

$$F(a, t) = 1 + at + N(0, x_0) \int_0^t f(s)ds, a \leq 0,$$

where  $f(x) = e^{Cx} - 1$ ,  $x \geq 0$ . It is easy to see that

$$\min_{t \geq 0} F(a, t) = F(a, g(-\frac{a}{N(0, x_0)})) = 1 + ag(-\frac{a}{N(0, x_0)}) + N(0, x_0) \int_0^{g(-\frac{a}{N(0, x_0)})} f(s)ds \triangleq G(a),$$

where  $g(x) = \frac{1}{C} \log(x + 1)$ ,  $x \geq 0$ , is the inverse function of  $f$ . Differentiating  $G(a)$  with respect to  $a$ , we obtain

$$\begin{aligned} \frac{d}{da}G(a) &= g(-\frac{a}{N(0, x_0)}) - g'(-\frac{a}{N(0, x_0)})\frac{a}{N(0, x_0)} + g'(-\frac{a}{N(0, x_0)})\frac{N(0, x_0)}{N(0, x_0)}\frac{a}{N(0, x_0)} \\ &= g(-\frac{a}{N(0, x_0)}) > 0, \quad a < 0. \end{aligned}$$

Notice that

$$\lim_{a \rightarrow -\infty} g(-\frac{a}{N(0, x_0)}) = +\infty.$$

Thus, we deduce that

$$\lim_{a \rightarrow -\infty} G(a) = -\infty,$$

which, together with that fact that  $G(0) = 1$  and the continuity of  $G$ , yields that there exists a unique  $a_0 < 0$  satisfies  $G(a_0) = 0$ . Therefore,  $G(a) \leq 0$  if  $a \leq a_0$ . Combining this with (4.22), if  $Q_x(0, x_0) \leq a_0$ , we may find a time  $0 < T_0 \leq g(-\frac{Q_x(0, x_0)}{N(0, x_0)})$  such that

$$e^{\int_0^t \inf_{x \in \mathbb{R}} Q_x(\tau, x) d\tau} \rightarrow 0, \text{ as } t \rightarrow T_0,$$

which, implies that

$$\liminf_{t \rightarrow T} \inf_{x \in \mathbb{R}} Q_x(t, x) \rightarrow -\infty, \text{ as } t \rightarrow T_0.$$

Therefore, in view of Theorem 4.1, we conclude that the solution  $(m, n)$  blows up at the time  $T_0$ .

**Theorem 4.4.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.1). Set  $Q(t, x) = \frac{1}{2}(u - u_x)(v + v_x)(t, x)$ . Assume that  $m_0, n_0 \in L^1$ , and that there exists some  $x_0 \in \mathbb{R}$  such that  $N(0, x_0) = |m_0(x_0)| + |n_0(x_0)| > 0$  and  $Q_x(0, x_0) = \frac{1}{2}(m_0(v_0 + v_{0x}) - n_0(u_0 - u_{0x}))(x_0) \leq -(2CN(0, x_0))^{\frac{1}{2}}$ . Then there exists a constant  $C = C(\|m_0\|_{L^1}, \|n_0\|_{L^1})$  such that the solution  $(m, n)$  blows up at a time  $T_0 \leq \frac{-Q_x(0, x_0)}{CN(0, x_0)}$ .*

Proof. In view of Lemma 4.6, we obtain that

$$Q_{xt}(t, x) + (Q(Q_x)_x)(t, x) + Q_x^2(t, x) \leq C(|m| + |n|)(t, x).$$

By Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned} & \frac{d}{dt} Q_x(t, q(t, x)) + Q_x^2(t, q(t, x)) \leq C(|m| + |n|)(t, q(t, x)) = C(|m_0(x)| + |n_0(x)|)q_x^{-1}(t, x) \\ & = CN(0, x) \exp\left(\int_0^t -\frac{1}{2}(m(v + v_x) - n(u - u_x))(\tau, q(\tau, x)) d\tau\right) \\ & = CN(0, x) \exp\left(\int_0^t -Q_x(\tau, q(\tau, x)) d\tau\right), \end{aligned}$$

form which it follows that

$$\frac{d}{dt} (Q_x(t, q(t, x)) \exp(\int_0^t Q_x(\tau, q(\tau, x)) d\tau)) \leq CN(0, x).$$

Integrating from 0 to  $t$  yields

$$\frac{d}{dt} \exp\left(\int_0^t Q_x(\tau, q(\tau, x)) d\tau\right) = Q_x(t, q(t, x)) \exp\left(\int_0^t Q_x(\tau, q(\tau, x)) d\tau\right) \leq CN(0, x)t + Q_x(0, x).$$

Integrating again from 0 to  $t$  yields

$$(e^{\int_0^t \inf_{x \in \mathbb{R}} Q_x(\tau, x) d\tau} \leq) \exp\left(\int_0^t Q_x(\tau, q(\tau, x)) d\tau\right) \leq \frac{1}{2}CN(0, x)t^2 + Q_x(0, x)t + 1.$$

Hence, if there exists some  $x_0 \in \mathbb{R}$  such that  $N(0, x_0) > 0$  and  $Q_x(0, x_0) \leq -(2CN(0, x_0))^{\frac{1}{2}}$ , then we may find a time  $0 < T_0 \leq \frac{-Q_x(0, x_0)}{CN(0, x_0)}$  such that

$$e^{\int_0^t \inf_{x \in \mathbb{R}} Q_x(\tau, x) d\tau} \rightarrow 0, \text{ as } t \rightarrow T_0,$$

which, implies that

$$\liminf_{t \rightarrow T} \inf_{x \in \mathbb{R}} Q_x(t, x) \rightarrow -\infty, \text{ as } t \rightarrow T_0.$$

Therefore, in view of Theorem 4.1, we conclude that the solution  $(m, n)$  blows up at the time  $T_0$ .  $\square$

**Remark 4.1.** We mention that, if  $v = 2u$ , Theorem 4.3 is same as Theorem 5.2 and Theorem 5.3 in [25], while Theorem 4.4 represents a new blow-up result for Eq.(1.5).

$$4.2. \quad N = 1, H = -\frac{1}{2}(uv - u_x v_x)$$

#### 4.2.1 A precise blow-up scenario

For  $N = 1$  and  $H = -\frac{1}{2}(uv - u_x v_x)$ , Eq.(1.1) is reduced to the following system:

$$(4.23) \quad \begin{cases} m_t + \frac{1}{2}((uv - u_x v_x)m)_x - \frac{1}{2}(uv_x - vu_x)m = 0, \\ n_t + \frac{1}{2}((uv - u_x v_x)n)_x + \frac{1}{2}(uv_x - vu_x)n = 0, \\ (m, n)|_{t=0} = (m_0, n_0), \end{cases}$$

where  $m = u - u_{xx}$  and  $n = v - v_{xx}$ .

Along the same lines as the proof of Lemma 4.1-4.2 and Theorem 4.1, we can obtain the following results.

**Lemma 4.8.** Let  $m_{10}, m_{20} \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $M = (m_1, m_2)$  to Eq.(4.23). Then the following system

$$(4.24) \quad \begin{cases} q_t(t, x) = \frac{1}{2}(uv - u_x v_x)(t, q), \quad t \in [0, T], \\ q(0, x) = x, \quad x \in \mathbb{R}. \end{cases}$$

has a unique solution  $q \in C^1([0, T] \times \mathbb{R}; \mathbb{R})$ . Moreover, the mapping  $q(t, \cdot)$  ( $t \in [0, T]$ ) is an increasing diffeomorphism of  $\mathbb{R}$ , with

$$(4.25) \quad q_x(t, x) = \exp\left(\int_0^t \frac{1}{2}(mv_x + nu_x)(\tau, q(\tau, x))d\tau\right).$$

---

**Lemma 4.9.** *Let  $m_{10}, m_{20} \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $M = (m_1, m_2)$  to Eq.(4.1). Then, we have for all  $t \in [0, T)$ ,*

$$(4.26) \quad m(t, q(t, x))q_x(t, x) = m_0(x) \exp\left(\frac{1}{2} \int_0^t (uv_x - vu_x)(\tau, q(\tau, x)) d\tau\right),$$

$$(4.27) \quad n(t, q(t, x))q_x(t, x) = n_0(x) \exp\left(-\frac{1}{2} \int_0^t (uv_x - vu_x)(\tau, q(\tau, x)) d\tau\right).$$

**Theorem 4.5.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.23). Then the solution  $(m, n)$  blows up in finite time if and only if*

$$\liminf_{t \rightarrow T} \inf_{x \in \mathbb{R}} ((mv_x + nu_x))(t, x) = -\infty \quad \text{or} \quad \limsup_{t \rightarrow T} \|(uv_x - vu_x)(t, \cdot)\|_{L^\infty} = +\infty.$$

#### 4.2.2 Blow-up phenomena

Now we derive four useful conservation laws for Eq.(4.23).

**Lemma 4.10.** *Let  $m_0, n_0 \in H^s$  with  $s > \frac{1}{2}$ , and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.23). Then we have that for all  $t \in [0, T)$ ,*

$$\begin{aligned} \int_{\mathbb{R}} (mv_x)(t, x) dx &= \int_{\mathbb{R}} m_0 v_{0x} dx, \quad \int_{\mathbb{R}} (nu_x)(t, x) dx = \int_{\mathbb{R}} n_0 u_{0x} dx, \\ \int_{\mathbb{R}} (mv)(t, x) dx &= \int_{\mathbb{R}} m_0 v_0 dx, \quad \int_{\mathbb{R}} (nu)(t, x) dx = \int_{\mathbb{R}} n_0 u_0 dx. \end{aligned}$$

Proof. By Eq.(4.1), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} (mv_x)(t, x) dx &= \frac{d}{dt} \int_{\mathbb{R}} (-nu_x)(t, x) dx \\ &= \int_{\mathbb{R}} (v_x m_t - u_x n_t)(t, x) dx \\ &= \frac{1}{2} \int_{\mathbb{R}} [(uv - u_x v_x)(mv_{xx} - nu_{xx})(t, x) + (uv_x - vu_x)(v_x m + u_x n)] dx \\ &= \frac{1}{2} \int_{\mathbb{R}} \partial_x ((uv - u_x v_x)(uv_x - vu_x))(t, x) dx = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} (mv)(t, x) dx &= \frac{d}{dt} \int_{\mathbb{R}} (nu)(t, x) dx = \int_{\mathbb{R}} (m_t v + n_t u)(t, x) dx \\ &= \frac{1}{2} \int_{\mathbb{R}} ((uv - u_x v_x)(mv_x + nu_x) + (uv_x - vu_x)(mv - nu))(t, x) dx \\ &= \frac{1}{2} \int_{\mathbb{R}} ((uv - u_x v_x) \partial_x (uv - u_x v_x) - (uv_x - vu_x) \partial_x (uv_x - vu_x))(t, x) dx = 0. \end{aligned}$$

This completes the proof of the lemma. □



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**Lemma 4.11.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.23). Assume that  $m_0$  and  $n_0$  do not change sign. Then there exists a constant  $C = C(\|v_{0x}m_0\|_{L^1}, \|v_0m_0\|_{L^1}, \|u_{0x}n_0\|_{L^1}, \|u_0n_0\|_{L^1}, \|u_0\|_{H^1}, \|v_0\|_{H^1})$  such that*

$$(4.28) \quad |u_x(t, x)| \leq |u(t, x)|, \quad |v_x(t, x)| \leq |v(t, x)|,$$

$$(4.29) \quad \|u(t)\|_{H^1} + \|v(t)\|_{H^1} \leq Ce^{Ct}, \quad \forall t \in [0, T).$$

Proof. Without loss of generality, we assume that  $m_0 \geq 0, n_0 \geq 0$ . Repeating the arguments that were used in Lemma 4.5, we get that the inequalities (4.11)-(4.18) still hold true here. Next, according to Lemma 4.10 with  $m, u + u_x, n, v - v_x \geq 0$ , we obtain

$$(4.30) \quad \begin{aligned} \|(mv_x)(t)\|_{L^1} &\leq \|(m(v - v_x))(t)\|_{L^1} + \|(mv)(t)\|_{L^1} \\ &= \int_{\mathbb{R}} (m(v - v_x))(t, x)dx + \int_{\mathbb{R}} (mv)(t, x)dx \\ &= 2 \int_{\mathbb{R}} (mv)(t, x)dx - \int_{\mathbb{R}} mv_x(t, x)dx \\ &\leq 2\|(m_0v_0)\|_{L^1} + \|(m_0v_{0x})\|_{L^1}. \end{aligned}$$

Finally, from Eq.(4.23), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2) &= \int_{\mathbb{R}} (m_t u + n_t v)(t, x)dx \\ &= \frac{1}{2} \int_{\mathbb{R}} ((uv - u_x v_x)mu_x + (uv_x - vu_x)mu \\ &\quad + (uv - u_x v_x)nv_x - (uv_x - vu_x)nv)(t, x)dx \\ &= \frac{1}{2} \int_{\mathbb{R}} ((u^1 - u_x^2)v_x m + (v^2 - v_x^2)u_x n)(t, x)dx \\ &\leq \frac{1}{2} (\|(u^2 - u_x^2)(t)\|_{L^\infty} \|(mv_x)(t)\|_{L^1} + \|(v^2 - v_x^2)(t)\|_{L^\infty} \|(nu_x)(t)\|_{L^1}) \\ &\leq C(\|u(t)\|_{L^\infty}^2 + \|v(t)\|_{L^\infty}^2) \leq C(\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2). \end{aligned}$$

Then the Gronwall lemma yields the desired inequality (4.29). This completes the proof of the lemma.  $\square$

**Theorem 4.6.** *Let  $m_0, n_0 \in H^s$  ( $s > \frac{1}{2}$ ), and let  $T > 0$  be the maximal existence time of the corresponding solution  $(m, n)$  to Eq.(4.23). Set  $Q(t, x) = \frac{1}{2}(uv - u_x v_x)(t, x)$ . Assume that  $m_0, n_0$  do not change sign, and that there exists some  $x_0 \in \mathbb{R}$  such that  $N(0, x_0) = |m(0, x_0)| + |n(0, x_0)| > 0$  and  $Q_x(0, x_0) = \frac{1}{2}(m_0 v_{0x} + n_0 u_{0x})(x_0) \leq a_0$ , where  $a_0$  is the unique negative solution to the following equation*

$$1 + ag\left(-\frac{a}{N(0, x_0)}\right) + N(0, x_0) \int_0^{g\left(-\frac{a}{N(0, x_0)}\right)} f(s)ds = 0,$$

with  $f(x) = \exp(e^{Cx} - 1) - 1$ ,  $x \geq 0$ ,  $g(x) = \frac{1}{C} \log(\log(x + 1) + 1)$ ,  $x \geq 0$ .

Then the solution  $(m, n)$  blows up at a time  $T_0 \leq g\left(-\frac{Q_x(0, x_0)}{N(0, x_0)}\right)$ .

Proof. It follows from Eq.(4.23) that

$$\begin{aligned}
(4.31) \quad & Q_{xt} + Q(Q_x)_x + Q_x^2 \\
&= - (1 - \partial_x^2)^{-1} \left( \partial_x(Q_x u) + (Q_x u_x) - \partial_x \left( \frac{1}{2} (uv_x - uv_x)m \right) \right) n \\
&\quad - (1 - \partial_x^2)^{-1} \left( \partial_x(Q_x v) + (Q_x v_x) + \partial_x \left( \frac{1}{2} (uv_x - uv_x)n \right) \right) m \\
&\quad + \frac{1}{2} (uv_x - vu_x)(mv_x - nu_x).
\end{aligned}$$

Using Lemma 4.11, and following along the same lines as the proof of Lemma 4.7, we obtain that

$$Q_{xt}(t, x_0) + (Q(Q_x)_x)(t, x_0) + Q_x^2(t, x_0) \leq C e^{Ct} (|m| + |n|)(t, x_0).$$

By Lemma 4.9, we get

$$\begin{aligned}
& \frac{d}{dt} Q_x(t, q(t, x_0)) + Q_x^2(t, q(t, x_0)) \leq C e^{Ct} (|m| + |n|)(t, q(t, x_0)) \\
& \leq C e^{Ct} N(0, x_0) \exp \left( \int_0^t -\frac{1}{2} (m(v + v_x) - n(u - u_x))(\tau, q(\tau, x_0)) d\tau \right) \exp \left( \frac{1}{2} \int_0^t \|(uv_x - vu_x)(\tau)\|_{L^\infty} d\tau \right) \\
& = C e^{Ct} N(0, x_0) \exp \left( \int_0^t -Q_x(\tau, q(\tau, x_0)) d\tau \right) \exp \left( \frac{1}{2} \int_0^t \|(uv_x - vu_x)(\tau)\|_{L^\infty} d\tau \right).
\end{aligned}$$

Again using Lemma 4.11, we have

$$\exp \left( \frac{1}{2} \int_0^t \|(uv_x - vu_x)(\tau)\|_{L^\infty} d\tau \right) \leq \exp \left( C \int_0^t e^{C\tau} d\tau \right) = \exp(e^{Ct} - 1),$$

from which it follows that

$$\frac{d}{dt} (Q_x(t, q(t, x_0)) \exp \left( \int_0^t Q_x(\tau, q(\tau, x_0)) d\tau \right)) \leq C e^{Ct} \exp(e^{Ct} - 1) N(0, x_0).$$

Integrating from 0 to  $t$  yields

$$\begin{aligned}
\frac{d}{dt} e^{\int_0^t Q_x(\tau, q(\tau, x_0)) d\tau} &= e^{\int_0^t Q_x(\tau, q(\tau, x_0)) d\tau} Q_x(t, q(t, x_0)) \leq Q_x(0, x_0) + N(0, x_0) \int_0^t \exp(e^{C\tau} - 1) C e^{C\tau} d\tau \\
&= Q_x(0, x_0) + N(0, x_0) (\exp(e^{Ct} - 1) - 1).
\end{aligned}$$

Integrating again from 0 to  $t$  yields

$$(4.32) \quad (e^{\int_0^t \inf_{x \in \mathbb{R}} Q_x(\tau, x) d\tau} \leq) e^{\int_0^t Q_x(\tau, q(\tau, x_0)) d\tau} \leq 1 + Q_x(0, x_0)t + N(0, x_0) \int_0^t (\exp(e^{C\tau} - 1) - 1) d\tau.$$

Next, following along almost the same lines as in the proof of Lemma 4.3 with  $f(x) = \exp(e^{Cx} - 1) - 1$ ,  $x \geq 0$  and  $g(x) = \frac{1}{C} \log(\log(x + 1) + 1)$ ,  $x \geq 0$ , completes the proof of the theorem.  $\square$

**Remark 4.2.** We mention that Theorem 4.6 is an improvement of Theorem 4.3 in [36]. Firstly, in [36] the authors assumed that  $\|u\|_{L^\infty}, \|v\|_{L^\infty} \leq C e^{Ct}$ , while in our paper,  $\|u\|_{L^\infty}, \|v\|_{L^\infty} \leq C e^{Ct}$  is ensured by Lemma 4.11. Secondly, in [36]  $x_0$  is required to satisfy an additional restriction:  $Q_x(0, x_0) = \inf_{x \in \mathbb{R}} Q_x(0, x)$ . Finally,  $a_0$  in our result is more explicit and accurate than that in [36].

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